Using a Modified Form of Lesson Study to Develop Students' Relational Thinking in Years 4, 5 & 6

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A modified form of Lesson Study was used to deliver several lessons over two weeks to develop students' relational thinking and to improve teachers' knowledge of this thinking. Fifteen students in a multi-grade 4, 5 & 6 classroom were surveyed and interviewed using true/false sentence and open number problems involving one unknown number before and after the study. Students in all three grades increased their understanding of the role of equivalence and their capacity to use relational thinking. Questionnaires were also undertaken by three participating teachers before and after the study. Their knowledge about students' relational thinking improved, and they demonstrated how they would integrate it into their future teaching of number and number operations.

Research has shown that the transition from arithmetic to algebra is difficult for many students. Carpenter, Franke and Levi (2003) found that many students perceive arithmetic only as a series of calculations. Students often do not see the relationships between numbers and operations when they carry out calculations. With increasing focus on the development of algebraic reasoning, many researchers advocate a closer integration of number and algebra in the primary school curriculum. The Australian Curriculum: Mathematics (ACARA, 2010) also aims to strengthen the links between the teaching of Number and Algebra, especially to the middle and later primary years.

This study used a modified form of Lesson Study as a research strategy to deliver lessons to build students' relational thinking and as a professional development platform for developing teachers' content knowledge by identifying the links between relational thinking and teaching arithmetic. This paper investigates the performance of both students and teachers on questionnaires before and after the Lesson Study and uses student interview data. It aims to shed light on the following research questions:

- What kind of thinking did students use to solve number sentences before the study?
- How can a modified form of Lesson Study to be used to move forward the development of students' relational thinking?
- What advantages did the students see in using relational thinking? How confident were they in thinking relationally across the four operations?
- What did the participating teachers learn about their students' relational thinking and will this inform their teaching and capacity to introduce relational thinking?

Current Research on Relational Thinking

Stephens (2006) and Hunter (2007) argue that relational thinking depends on treating the equal sign as an indicator of equivalence, whereas many primary school students treat the equal sign simply as a command or direction to find the answer. Carpenter, Levi, Franke, and Zeringue (2005) provide an example used in their study of Grade 2 and 3 students, where some students gave 12 as a response to the missing number sentence 8 + 4 = __ + 5; and other students gave 17, by adding all the numbers 8 + 4 + 5. Hunter (2007) along with Stephens (2006a), Molina and Ambrose (2008) also point out that students'

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inadequate understanding of the equality symbol leads to difficulties in solving symbolic expressions and equations.

Stephens and Ribeiro (2012) and Irwin and Britt (2005) identify relational thinking in the methods of equivalence and compensation that some students use in solving number sentences; for example, by transforming 17 + 68 into 15 + 70 by adding 2 to 68 and subtracting 2 from 17. Researchers, such as Carpenter, Franke, and Levi (2003), Hunter (2007), Molina and Ambrose (2008), and Stephens (2006a) suggest that relational thinking can be fostered by posing true/false and open number sentences, and assisting students to focus on the sentence as a whole rather than as a computation. For example, Carpenter and Franke (2001) asked students if 78 – 49 + 49 = 78 was true or false, and how did they decide; looking if students could refrain from computation and affirm that it was true by attending to the mathematical structure of the sentence. Investigating true/false number sentences played a key role in this action research study.

Jacobs, Franke, Carpenter, Levi, and Battey (2007) also used true/false and open number sentences as contexts in which teachers could orchestrate conversations with students in order to identify the kind of thinking embedded in students' strategies in deciding whether particular number sentences were true or false, and the methods they used to solve missing number sentences involving the four operations.

Following Jacobs et al. (2007), this study also aimed to enhance teachers' content knowledge of students' relational thinking; in particular, how that reasoning supports students' understanding of number and number operations. In helping to plan, implement and evaluate the research lessons, participating teachers were able to focus on using equivalence and number relations to simplify calculations. Participating teachers interviewed students to elicit their strategies and to assess students' relational thinking. Teachers' content knowledge of relational thinking was also assessed by presenting them with students' responses to various open number sentences.

Method

Four teachers and fifteen students from a rural primary school in western Victoria participated in this research. The participants included the school principal, one teacher from Prep/Grade 1, one teacher from Grades 2/3, the teacher researcher from Grades 4/5/6; and fifteen students from Grades 4/5/6, aged between 9 and 12.

Lesson Study as research

The usual aim of the Lesson Study is the professional development of the teachers: how to improve teachers' understanding of what students learn and how best to bring that about (Fernandez and Yoshida, 2004). All teachers research and plan a lesson. The lesson is then taught by one of the teachers in the group with the others observing, perhaps with some visitors. In a debriefing session, the lesson design and its implementation are analysed, with a special focus on how well the students were able to demonstrate what they were intended to learn. Modifications to the lesson are then made to better achieve the intended learning goals. In the next cycle, the revised lesson is taught to a different class by another teacher, and that lesson is then followed by further review and modification where necessary.

This study adopted the approach used by Pierce and Stacey (2009) where Lesson Study is used *for research* involving students and teachers, as well as for teachers' professional development. Two lessons were researched and delivered by the teacher researcher to a multi-grade 4, 5 and 6 over two weeks. Evidence was collected before, during and after the lessons from students and teachers as discussed below.

Space does not permit a detailed elaboration of the two lessons. Key goals of the lessons were to have students become familiar with number sentences with more than one number following the equal sign; to understand the equal sign means 'same as'; and thus show how to balance equations involving the four operations by using number facts and computations, thus opening up basic properties of relational thinking. Students were asked to identify true or false number sentences; to solve missing numbers sentences and to share the strategies they used; and to create some sentences of their own where relational thinking can simplify calculations. A key question for the research was whether were students able to identify the variation between numbers on both sides of equal sign, and to show in their solutions that the direction of variation depends on the operation involved. The study of Molina and Ambrose (2008) was used to categorise those students who had misconceptions in regard to the equal sign, and shifts in other students' thinking from relying solely on computation, to using a mix of computational and relational strategies, to relying only on r elational thinking. Participating teachers observed the lessons, recorded students' mathematical thinking, and debriefed following the lessons. The post-lesson debriefing was used to fine tune the delivery of the second lesson. Interviews with students were conducted by the teacher researcher and by one of the participating teachers.

Student and teacher questionnaires

Questionnaires before and after the Lesson Study were administered to assess students' capacity for relational thinking and to identify any improvement as a result of the Lesson Study. Observation checklists were used by participating teachers during the lessons in order to gather more evidence of students' relational thinking. Interviews with students were also used to confirm the evidence obtained from these sources.

Two student questionnaires were administered before the Lesson Study. The Grade 4 questionnaire had six missing number questions involving all four operations, such as 33 + 19 = __ + 20; __ + 17 = 15 + 24; 78 – 39 = __ – 40; 14 × 5 = 7 × __; 12 ÷ 4 = __ ÷ 2. Grade 5/6 students were given eight missing number questions with some involving larger numbers, such as 199 + 271 = 200 + __; 137 – 98 = __ – 100. Students were invited to explain how they had worked out the answer for each question. Each questionnaire also included True/False questions. For Grade 4, these included 27 + 48 – 48 = 27 (T/F); 15 + 19 = 15 + 20 – 1 (T/F); 99 – 9 = 90 – 59 (T/F); 3 × 4 = 12 × 2 (T/F); 18 ÷ 6 = 6 ÷ 2. Students were asked in each case why they had chosen True or False.

Two similar questionnaires were given after the Lesson study. The one for Grade 4 and some Grade 5/6 students included open number sentences involving addition, subtraction and multiplication. The one for Grades 5/6 involved all four operations with relatively larger numbers that used in the Grade 4 v ersion. The results of the questionnaires and interviews are summarised in Table 1 below.

Before the first lesson, a questionnaire was conducted to identify teachers' knowledge of relational thinking and their experience in teaching relational thinking. Teachers were asked to solve similar missing number sentences as for Grades 5/6 and to explain how they had solved each problem. They were also asked to identify possible misconceptions of students in attempting to solve missing number sentences; and whether they had previously used relational thinking in their teaching. After the Lesson Study, teachers were given another set of missing number questions; and were asked whether they intended to incorporate relational thinking in their future teaching. The questionnaire also re-examined teachers' understanding of students' possible misconceptions of the equal sign.

Results

Table 1 shows students' use of relational thinking before and after the Lesson Study. The four levels are based on the categories used by Molina and Ambrose (2008). Level 1 students display misconceptions in relation to equal sign; Level 2 students use only computation to solve open number sentences; Level 3 students use a mix of computation and relational thinking to solve problems and Level 4 students use only relational thinking.

In Grade 4, all three students moved from Level 1 to Level 3. In Grade 5, one student who was absent for one of the lessons remained at Level 1, but the other three all moved to Levels 3 and 4. Grade 6 students who were at Level 2 before the Lesson Study all moved to Levels 3 and 4. One Grade 6 remained at Level 4.

Table 1
Relational thinking before and after the Lesson Study

Grade	Level 1		Level 2		Level 3		Level 4	
	Before	After	Before	After	Before	After	Before	After
Grade 4 (N=3)	3	0	0	0	0	3	0	0
Grade 5 (N=4)	2	1	1	0	1	1	0	2
Grade 6 (N=8)	2	0	4	1	1	3	1	4

Students' Understanding of Equal Sign

Before the Lesson Study seven students had misconceptions in relation to equal sign; three from Grade 4, two from Grade 5 and two from Grade 6. Some students wrote 52 for the question 33 + 19 = ___ + 20 because 33 + 19 = 52 disregarding the 20 on the right side. Others wrote 72 because 33 + 19 + 20 = 72. None of the above students could find the missing number for the question ___ + 17 = 15 + 24, as one explained, "You cannot plus anything from 17 to make 15." Most of these showed misconceptions in true/false questions. For example, they circled True for the problem 99 – 9 = 90 – 59, giving a reason that 99 – 9 = 90. One student added 34 + 28 + 30 + 20 + 4 + 8 = 134 (*should be 124*) for the true/false question 34 +28 = 30 + 20 + 4 + 8. These students all appeared to treat the equal sign as a command to give an answer to the operations expressed on the left side, or even both sides, of the equal sign.

Interestingly, one such Grade 4 student Athena gave a correct response and reasoning for the true/false question 99 – 9 = 90 – 59 even though she had misconceptions in the first part of the questionnaire (see below). She worked out the left side 99 – 9 = 90 and the right side 90 – 59 = 31 and chose false. Among most of the true/false questions, Athena started working out the value of left side and right side by using an algorithm. During the interview, she said, "When I was working the question 34 + 28 = 30 + 20 + 4 + 8, I found 34 + 28 = 62 so is 30 + 20 + 4 + 8 = 62." She added, "I also know my times-tables very well, for the question 3 × 4 = 12 × 2, I know 3 × 4=12 and 12 × 2 = 24 so it's false." Athena's grasp of number facts helped her to compute the value of both sides of equal sign and develop her understanding of equivalence, which was confirmed during the interview. After the interview, most students understood that the equal sign means 'the same as' or "is equivalent to”. However, as mentioned above, one Grade 5 student who was absent in one of Lesson Study sessions still displayed misconceptions in relation to equal sign.

Students’ Perceptions of Relational Thinking

In the questionnaire before the Lesson Study, Athena (Grade 4) wrote 33 + 19 = 52 + 20 disregarding 20 on the right side. In another problem, __ + 17 = 15 + 24 she wrote, “I don’t know because you can’t plus 17 to make 15.” As mentioned earlier, Athena changed her understanding of the equal sign during the true/false questions, using computational strategies to balance both sides. In the questionnaire after the Lesson Study, Athena used a mix of computational and relational thinking. In her answer to the question __ + 16 = 15 + 24, Athena explained, “Between 16 and 15 is +1 so between 24 and 23 is –1 because 16 is a big number you need a small number to keep it balanced.” However, she did not use relational thinking in the multiplication question 18 × 5 = 9 × __; instead, she did 18 × 5 = 90 and 90 ÷ 9 = 10.

Troy (Year 5) also displayed various misconceptions in relation to equal sign in the questionnaire before the Lesson Study; writing incorrectly that 33 + 19 = 52 + 20 and also 78 – 39 = 41 – 40. After the Lesson Study, Troy successfully demonstrated his understanding of relational thinking in addition and multiplication equations. In correctly solving 13 + 29 = 12 + 30 Troy explained that 29 + 1 makes 30 and 13 – 1 makes 12. In the multiplication question, 18 × 5 = 9 × 10, he wrote that 18 ÷ 2 = 9 so 5 × 2 = 10. However, he was confused with the direction of compensation in subtraction questions. For example, in incorrectly giving 71 – 28 = 73 – 26. He wrote that 71 + 2 = 73 so 28 – 2 = 26.

In the questionnaire before the Lesson Study, Joe (Year 6) used a mix of computational and relational thinking strategies to solve problems. For example, in 18 + 29 = ___ + 30, Joe wrote that 18 + 29 = 47, but he also used arrows vertically to show 29 plus 1 makes 30 and 18 minus 1 makes 17, therefore 17 + 30 = 47. In his response for the problem ___ – 38 = 75 – 40, Joe wrote that 75 – 40 = 35, then he used arrows vertically to show 40 minus 2 makes 38 and 75 minus 2 makes 73, therefore 73 – 38 = 35. For the multiplication problem 48 × 25 = ___ × 100, Joe worked out the left side of equal sign 48 × 25 = 1200 then he calculated right side that 12 × 100 = 1200. For division problem, 24 ÷ 6 = ___ ÷ 3, Joe worked out the left side of equal sign 24 ÷ 6 = 4, then right side 4 × 3 = 12 so 12 ÷ 3 = 4. Dealing with true/false questions, for example, with 570 + 199 = 570 + 200 – 1, Joe circled True and explained, “200 – 1 = 199.” After the Lesson Study, Joe used arrows from left side to right side to show one direction and variation of compensation; then he used the correct direction of variation between the uncalculated equations on each side of the equal sign to solve the problem. He confidently used relational thinking and correctly solved four operations without carrying out any computation to check the answer.

In the pre-Lesson Study questionnaire, Emma (Year 6) used arrows to show the directions of compensation, consistently directing the arrow from left to right no matter where the missing number was. For 18 + 29 = ___ + 30, she explained, “29 plus 1 makes 30 so 18 minus 1 makes 17 to keep it balanced.” In another question, ___ + 17 = 15 + 24, Emma explained, “17 minus 2 makes 15, so 24 minus 2 makes 22, therefore you don’t have two big numbers on the same side.” In the true/false question, 570 + 199 = 570 + 200 – 1, Emma explained, “It’s true because it’s just split so it is easier.” In another question, 12 × 6 = 72 × 6, she explained, “It’s false because 72 and 12 are very different numbers.” Emma’s responses demonstrate that she looked at the equation as a whole and simplified the calculation. In her post-Lesson Study questionnaire responses, Emma not only worked out the variations between two numbers but also the directions of compensation. For example, in the question, ___ + 199 = 152 + 200 Emma used arrows from 200 minus 1 to 199 and from 152 plus 1 to 153 and explained, “When you plus one you have to take away one so it stays balanced.” In a later subtraction question, 12.8 – 3.2 = __ – 3, Emma used arrows from 3.2 minus 0.2 to 3 and from 12.8 minus 0.2 to 12.6 and she explained, “The difference of numbers has to stay the same.” In her answer for the multiplication problem 36 × 25 = 9 × ___, Emma showed that 36 divided by 4 makes 9 so 25 times by 4 makes 100 and she explained, “You can’t have 2 big numbers on the same side.” In her answer for the division problem ___ ÷ 15 = 20 ÷ 5, Emma showed that 5 times 3 makes 15 so 20 times 3 makes 60 and she explained, “The gap or difference between two numbers has to stay the same.” Before and after the Lesson Study, Emma applied clear relational thinking in open number sentence problems involving addition, subtraction, multiplication and division; not needing to use any algorithm to check her answers like many other students.

Teachers’ Understanding of Students’ Misconceptions of Equal Sign

In the questionnaire before the first lesson, teachers were asked to give all possible student responses to the missing number sentence 15 + 8 = __ + 10. All teachers gave 13 as a possible student response for the question, reasoning that since 15 + 8 = 23, so 23 – 10 = 13. Only one teacher gave 33 as a possible student response, explaining that students might think 15 + 8 + 10 = 33. No teacher pointed to 23 as a possible misconception. Before the Lesson study, teachers’ awareness that students may treat the equal sign as meaning ‘the answer comes next’ appeared to be limited.

After the Lesson Study, teachers demonstrated better understanding of students’ misconceptions in relation to the equal sign. According to their questionnaire responses, all three teachers pointed out the correct answer 24 and the wrong answer 34 for the sentence 25 + 9 = ___ + 10. They explained that students might do 25 + 9 = 34 disregarding the number 10 on the right side of equal sign. Two teachers also pointed out that 26 could be a possible answer for some students who may use relational thinking but did not work out the correct direction of variation. One also wrote 44 because students may add all the numbers up, 25 + 9 + 10 = 44. After the Lesson Study, teachers showed improved understanding of students’ misconceptions of equal sign. However, only one teacher recognised all the possible misconceptions that students might make.

Teachers’ Knowledge of Relational Thinking

None of participating teachers had explicitly taught relational thinking before the Lesson Study, but they appeared to know that relational thinking relates to equivalence problems. In solving missing number sentences involving addition and subtraction, they did use relational thinking, but all used computational strategies to solve multiplication and division problems, such as 48 × 25 = ___ × 100, and ___ ÷ 15 = 20 ÷ 5.

Throughout the two cycles of Lesson Study, participating teachers expanded their understanding of relational thinking, particularly in discussing students’ responses after the two lessons. During the lessons, teachers had opportunities to explore students’ strategies embedded in their solutions and to orchestrate discussion with some of students in relation to equal sign, computation and the direction of variation they used.

In the post-Lesson Study questionnaire, teachers were all able to use relational thinking to solve multiplication and division problems, and referred to the advantages of relational thinking might have over a purely computational approach. They agreed that students need to know their number facts, including multiplication facts, and what the equal sign means before applying a relational approach.

Teachers' Intentions regarding Teaching Relational Thinking

Participating teachers studied students' responses and found that it is easier to start the direction of variation from the side where there is no missing number so students could identify the correct direction of variation between the uncalculated equations. After the Lesson Study, teachers all proposed to integrate relational thinking into their mathematics lessons. For example, the Prep/One teacher wrote, "I will try to teach students the near 10 and near double strategies so that students can see the links, for example, if 6 + 10 = 16, then 6 + 9 = ___." Another teacher wrote, "Establish a better understanding of the equal sign, e.g. 10 + 3 = ___ + ___." All teachers agreed that relational thinking could be used to simplify calculations and to check answers; and especially in simplifying calculations where addends or subtrahends can be rounded up, or down, to the nearest ten or hundred. All agreed to focus more on developing students thinking with regard to equivalence and compensation at whatever year level.

Discussion and Conclusion

This study provided evidence that Lesson Study focused on students' mathematical thinking in solving open number sentences was productive for both teachers and students. It clearly provided opportunities for teachers to discover students' misconceptions in relation to equal sign, and their need to pay attention to the sentence as a whole.

Before the Lesson Study, many students relied completely on calculation. According to Molina and Ambrose (2008), this behaviour is a result of the strong orientation to computation which dominates arithmetic in early years. True/false and open number sentences proved to be useful tools for seeding discussions about the equal sign and developing students' relational thinking. At the end of the Lesson Study, the number of students who were able to use relational thinking increased across all year levels. The most significant increase was evident among Grades 5 and 6 students. Many could solve number sentences using the four operations solely by using relational thinking. It is clear that fluency with number facts played a vital role in assisting these students to use relational thinking.

As a result of the Lesson Study, almost all students were able to use equivalence and the correct direction of compensation/variation between numbers to solve problems. However, some students still failed to identify the correct direction of compensation or variation. These responses confirmed findings by Irwin and Britt (2005) and Stephens (2006) that students need help to distinguish between the direction of compensation for addition and subtraction sentences. Students also need to know that direction of compensation is also different between multiplication and division.

Throughout the two Lesson Study cycles, participating teachers had opportunities to investigate relational thinking among Grades 4/5/6 students, and to observe students as they moved away from relying solely on computational strategies to using relational thinking in solving open number sentences. Focussing on the structure of number sentences, the operations involved, and the key ideas of equivalence and compensation are all necessary to strengthen links between number and algebra. Our study supports the findings of Carpenter et al. (2003), Hunter (2007), Molina and Ambrose (2008), Stephens (2006a) that solving and discussing true/false and open number sentence problems are effective ways to foster relational thinking. It also showed that fluent recall of number knowledge is a pre-condition of students' use of relational thinking across the four operations; and that relational thinking is influenced by the characteristics of the sentence, size and type of numbers used, and the operations involved.

Participation in Lesson Study improved teachers' understanding of the mathematics behind relational thinking strategies, and of children's misconceptions that need to be addressed directly. Teachers recognised the key role played by fluent recall of number facts to support the twin ideas of equivalence and compensation. They were all able to point to specific instances of how they could and would integrate relational thinking into their teaching of number and number operations at whatever grade level.

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